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Exact time evolution of the density of a classical many-body system: The open one-dimensional gravitational gas

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We introduce an exact equation for the time evolution of a classical many-body system, and apply it to the one-dimensional gravitational gas. An irreversible approach to a final density distribution is found for a large class of initial momentum distributions, allowing us to introduce the idea of ''semiergodicity.'' [S1063-651X(96)11407-0]

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I. INTRODUCTION

An incomplete statistical description of a classical many-body problem may be expressed by the time evolution of the one-particle distribution function $f(r,p,t)$, where r and p represent the coordinates and momentum of one particle. To complete the description, it would be desirable to specify the correlation functions as well, but given the severe difficulty of specifying the behavior of all N particles of a system, we choose to start first with $f(r,p,t)$. To achieve this goal, we adopt some formal results reported earlier [1–3] in both classical and quantum systems. Furthermore, in this work, we limit ourselves to a one-dimensional gas consisting of identical particles that interact by gravitation, the favorite toy system in stellar dynamics. We find that for some initial conditions, the system irreversibly approaches some spatial density distribution. We suggest to call such a system ''semiergodic''—ergodic only in spatial behavior. We also attempt to show that the formation of microstructures occurs quite naturally in the one-dimensional gravitational gas.

The mathematical method we will use is, to our knowledge, new even in nonequilibrium statistical mechanics, so we find it appropriate to quickly reexpress our method of iterated projections for application to stellar dynamics. So in Secs. II and III, we offer a summary of the procedure to arrive at an equation for the exact time evolution of the density of a system. After the formal presentation, we end up with a final formula that has no more bearing with the method of iterated projections, with wide applicability to many systems.

In Sec. IV, we reduce the formula to the case of one dimension, then we specialize to the gravitational pair potential in one dimension given by

$$V=\gamma|r_i-r_j|, \tag{1}$$

where γ is a constant for identical particles. We consider one initial condition representing the one-dimensional equivalent of a nonsingular ''petite-bang'' initial condition, which reduces in the appropriate limits to a singular ''petite bang'' [4], or ''petite collapse.'' This example is a ''cosmological'' must for any claimed exact time evolution equation. Having established the use of the exact equations for the one-dimensional gravitational gas, we offer in the last section some suggestions on the application of our general approach to three dimensions.

II. APPLICATION OF THE METHOD OF ITERATED PROJECTIONS

The process of contracting the description from the N-particle distribution function f_N^N [N spatial coordinates (subscript) and N momenta (superscript)] to the one-particle distribution function, $f(r,p,t)=f_1^1$, is accomplished in two steps. First, we derive an exact formal solution for $f_1^N(r,p^N,t)$, and then integrate over all momenta but those belonging to one particle, effected by the operator $I_1=\int dp^{N-1}$. The reason for doing this in two steps is that the integral over unwanted space coordinates may be defined as a projection operator, while integration over the momenta is not a projection operator.

Let us define

$$P_k=\frac{1}{\Omega^{N-k}}\int dr^{N-k}, \tag{2}$$

where Ω is the volume of the system. This volume will disappear from our final results once we properly normalize the distribution functions.

When $k=0$, P_o integrates out all space coordinates, and one gets, for example,

$$\phi_N(p^N)=P_o f_N, \tag{3}$$

the N-particle momentum distribution function. From Eq. (3), one can define

$$\varphi(p,t)=I_1\phi_N=\int dp^{N-1}\phi_N. \tag{4}$$

In the past, projection techniques have not resulted in numeric results, as may be verified by examining the literature [5]. By contrast, we carry out our operator formalism to produce numerical results. To show this, it is worthwhile first to present some formal results [1–3] applied to nonequilibrium statistical mechanics.

Following a method utilized in several forms [1–3], which goes well beyond the usual approach using projection techniques [5], one can write the formal solution for $P_k f_N$, which for now we simply write as $Pf_N = f_P$:

$$f_P = g + h, \quad g = \sum_{m=0}^{\infty} g_m, \quad h = \sum_{n-1}^{\infty} h_n, \tag{5}$$

where

$$g_0 = F(t,0) f_P(0), \quad h_1 = \int_0^t ds_1\, F(t,s_1) PLG(s_1,0) Q f_N(0), \tag{6}$$

$$g_m = \int_0^t ds_1 \int_0^{s_1} ds_2\, J(t,s_1,s_2) g_{m-2}(s_2), \tag{7}$$

$$h_n = \int_0^t ds_1 \int_0^{s_1} ds_2\, J(t,s_1,s_2) h_{n-2}(s_2), \tag{8}$$

$$J(t,s_1,s_2) = F(t,s_1) P_o (L_o + L_i) G(s_1,s_2) Q (L_o + L_i), \tag{9}$$

$$F(t,s_1) = e^{(t-s_t)PL}, \quad G(s_1,s_2) = e^{(s_1-s_2)QL}, \tag{10}$$

$$Q = 1 - P, \tag{11}$$

$$L = L_o + L_i,$$

$$L_o = -\sum_{j=1}^{N} \frac{p_j}{m} \frac{\partial}{\partial r_j},$$

$$L_i = \frac{1}{2} \sum_{j \neq k} \frac{\partial V(|r_j - r_k|)}{\partial r_j} \left(\frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_k} \right). \tag{12}$$

We now perform the following general procedure analogous to those used for quantum systems [1,3] but not detailed in the classical version [2]:

Step 1. First, observe that $G(s_1,s_2)QL = QLG(s_1,s_2)$ so

$$g_m = \int_0^t ds_1 \int_0^{s_1} ds_2\, F(t,s_1) PLQLG(s_1,s_2) g_{m-2}(s_2). \tag{13}$$

Step 2. Differentiate Eq. (13) two times:

$$g_m' = PLg_m + PLQL \int_0^t ds_2 G(t,s_2) g_{m-2}(s_2), \tag{14}$$

$$g_m'' = PLg_m' + PLQLg_{m-2} + PLQLQL \int_0^t ds_2 G(t,s_2) g_{m-2}(s_2)$$

or

$$g_m'' - [PL + PLQL(PL)^{-1}] g_m' - PLQLg_{m-2} + PLQLg_m = 0, \tag{15}$$

where if necessary, we use the Feynman definition for the inverse operator

$$(PL)^{-1} = \int_0^{\infty} d\omega\, e^{-\omega PL}, \tag{16}$$

Step 3. Sum Eq. (15) from $m=0$ to infinity to give

$$g'' - [PL + PLQL(PL)^{-1}] g' = g_0'' - [PL + PLQL(PL)^{-1}] g_0' + PLQLg_0 = 0, \tag{17}$$

where we have used the definition for g_0 in Eq. (6).

Step 4. Set the boundary conditions

$$g(0) = f_P(0), \qquad g'(0) = g_0'(0) = 0. \tag{18}$$

Step 5. Follow a similar procedure for h_n to give

$$h' = h_1' + PLh + PLQL \int_0^t ds_2 G(t,s_2) h(s_2), \tag{19}$$

$$h'' - [PL + PLQL(PL)^{-1}] h' = h_1'' - [PL + PLQL(PL)^{-1}] h_1' + PLQLh_1 = 0, \tag{20}$$

again using the definition in Eq. (8).

The fact that the inhomogeneous terms in Eqs. (17) and (20) are zero is a rigorous property for any P.

Step 6. Set the boundary conditions

$$h(0) = 0,$$

$$h'(0) = h_1(0) = PLf_Q(0). \tag{21}$$

Step 7. Evaluate

$$PL + PLQL(PL)^{-1} = PL + PL(1-P)L(PL)^{-1} = -PL = PL^2(PL)^{-1} \tag{22}$$

so that

$$g'' - PL^2(PL)^{-1} g' = 0 \quad \text{and} \quad h'' - PL^2(PL)^{-1} h' = 0. \tag{23}$$

With the boundary conditions given by Eqs. (18) and (21), the solutions of Eqs. (23) are

$$g = f_P(0) + [\exp(tK) - 1] K^{-1} PLf_N(0), \tag{24}$$

$$h = [\exp(tK) - 1] K^{-1} PL(1-P) f_N(0). \tag{25}$$

Let us put $P = P_1 = (1/\Omega^{N-1}) \int dr^{N-1}$. With this projection operator, we find that the exact solution for the function $f_{P_1}(t) = f_1^N(r_1, p^N, t)$ from Eq. (5) is given by

$$f_{P_1} = f(r, p^N, 0) + [\exp(tK) - 1] K^{-1} P_1 L f_N(0), \tag{26}$$

where $K = PL^2(PL)^{-1}$.

Integrating this over all momenta but one, we get

$$f(r,p,t)=f(r,p,0)+\int dp^{N-1}[\exp(tK)-1]K^{-1}P_1Lf_N(0), \tag{27}$$

which we will now simplify.

III. ONE-PARTICLE DISTRIBUTION FUNCTION

We first rewrite Eq. (27) as

$$f(r,p,t)=f(r,p,0)+I\sum_{j=1}^{\infty}\frac{t^jK^{j-1}}{j!}PLf_N^N(0), \tag{28}$$

whose sum we now decompose and simplify term by term. We follow the technique in Refs. [1–3], and arrive at some expressions for each term of the series. Although somewhat repetitive, it is important to present the details because the operator techniques used in this paper are uncommon in classical physics. $j=1$:

$$\frac{t}{1!}IPLf_N^N(0)=\frac{t}{1!}\left[L_o(1)f(r,p,0)+n_0\int dr'\,\frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}f_2^1(r,r',p,0)\right], \tag{29}$$

where $n_0=(N-1)/\Omega$, the average density for the system and $L_o(1)$ is now simply a one-particle expression. $j=2$:

$$\frac{t^2}{2!}IPL^2f_N^N(0)=\frac{t^2}{2!}[L_o^2(1)f(r,p,0)+IPMf_N^N(0)], \tag{30}$$

where $M=L_oL_i+L_iL_o+L_i^2$. $j=3$:

$$\frac{t^3}{3!}I(PL^2)(PL)^{-1}PL^2f_N^N(0)=\frac{t^3}{3!}I(PL^2)(PL)^{-1}[PL_o^2+PM]f_N^N(0). \tag{31}$$

We are now obliged to simplify $I(PL^2)(PL)^{-1}P$ since this expression will recur in later expressions. The rightmost P makes everything to the right of the expression a function of only one coordinate r, where we have suppressed the particle label 1. Now from Eq. (16),

$$\begin{aligned}(PL)^{-1}P&=\int_0^\infty d\omega\,\exp(-\omega PL)P\\&=\int_0^\infty d\omega\sum_{j=0}^{\infty}\frac{(-\omega PL)^j}{j!}P\\&=\int_0^\infty d\omega\sum_{j=0}^{\infty}\frac{(-\omega PL_o)^j}{j!}P=(PL_o)^{-1}P.\end{aligned} \tag{32}$$

Equation (32) is shown by using the fact that the integral of the force on a particle over all space is zero.

Next, we look at

$$\begin{aligned}PL^2P&=(PL_o^2+PL_oL_i+PL_iL_o+PL_i^2)P\\&=[L_o^2(1)+L_o(1)PL_i+PL_iL_o+PL_i^2]P.\end{aligned}$$

In the second term, we again encounter $PL_iP=0$ and in the third term L_o and P commute, to make it zero as well, allowing us to write

$$IPL^2P=I(PL_o^2+PL_i^2)P=\left(L_o^2(1)+b\frac{\partial^2}{\partial p^2}\right)IP,$$

$$b=\frac{(N-1)}{\Omega}\int dr'\left[\frac{\partial V(r-r')}{\partial r}\right]^2\cong n_0\int dr'\left[\frac{\partial V(r-r')}{\partial r}\right]^2, \tag{33}$$

where $n_0=(N-1)/\Omega$. b is in general a function of r and the location of the walls of the vessel, but when an infinite system is considered, it may turn out to be just a constant, as in analogous evaluations of similar expressions in virial expansions. We have not gone to the thermodynamic limit and we should remember this subtlety. Accordingly, we should not always treat it as a constant when operators act on it, but in this paper, all the operators we use are such that it is effectively a constant.

With the definition Eq. (33), we can rewrite Eq. (31) as

$$\frac{t^3}{3!}I(PL^2)(PL)^{-1}PL^2f_N^N(0)=\frac{t^3}{3!}[L_o(1)+b\nabla_p^2L_o^{-1}(1)]\times[IPL_o^2+IPM]f_N^N(0) \tag{34}$$

and from now on, we may write

$$IKP=IPL^2(PL)^{-1}P=[L_o(1)+b\nabla_p^2L_o^{-1}(1)]IP, \tag{35}$$

where we give the usual Feynman interpretation for the inverse operator.

Finally for $j=3$, we can write

$$\begin{aligned}&\frac{t^3}{3!}I(PL^2)(PL)^{-1}PL^2f_N^N(0)\\&=\frac{t^3}{3!}\{[L_o^3+b\nabla_p^2L_o]f(r,p,0)\\&+[L_o+b\nabla_p^2L_o^{-1}]IPMf_N^N(0)\},\end{aligned} \tag{36}$$

where we have now removed the index (1) for the free particle operator L_o as is now clear that all future occurrences of the operator refer to a single particle.

The expression $IPMf_N^N$ may still be simplified:

$$IPMf_N^N = IP(L_oL_i + L_iL_o + L_i^2)f_N^N(0) = L_oIPL_if_N^N + IPRf_N^N = -n_0 \frac{p}{m}\frac{\partial}{\partial r}\int dr' \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p} f_2^1(r,r',p,0)$$
$$-n_0\int dr' \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}\left(\frac{p}{m}\frac{\partial}{\partial r} f_2^1(r,r',p,0)\right) - n_0\int dp' \int dr' \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}\left(\frac{p'}{m}\frac{\partial}{\partial r'} f_2^2(r,r',p,p',0)\right)$$
$$+n_0^2\int dr' \int dr'' \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}\left(\frac{\partial V(r-r'')}{\partial r}\frac{\partial}{\partial p} f_3^1(r,r',r'',p,0)\right)$$
$$+n_0\int dr' \left(\frac{\partial V(r-r')}{\partial r}\right)\frac{\partial}{\partial p}\left(\frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p} f_2^1(r,r',r'',p,0)\right). \tag{37}$$

We write down the $j=4$ term to exhaust all the tricks that one could use, and introduce the notation $\Delta = b\nabla_p^2$:

$$\frac{t^4}{4!} IK^3PL = \frac{t^4}{4!} IKPK^2PL = \frac{t^4}{4!}(L_o + \Delta L_o^{-1})^2 IPK^2PLf_N^N(0) = \frac{t^4}{4!}(L_o + \Delta L_o^{-1})^2 IPL^2 f_N^N(0)$$
$$= \frac{t^4}{4!}\{[L_o^4 + L_o\Delta L_o + \Delta L_o^2 + \Delta L_o^{-1}\Delta L_o]f(r,p,0) + [L_o^2 + L_o\Delta L_o^{-1} + \Delta + \Delta L_o^{-1}\Delta L_o^{-1}]IPMf_N^N(0)\}$$
$$= \frac{t^4}{4!}\{[L_o^4 + L_o\Delta L_o + \Delta L_o^2 + \Delta L_o^{-1}\Delta L_o]f(r,p,0) + [L_o^3 + L_o\Delta + \Delta L_o + \Delta L_o^{-1}\Delta]IPL_if_N^N(0)$$
$$+ [L_o^2 + L_o\Delta L_o^{-1} + \Delta + \Delta L_o^{-1}\Delta L_o^{-1}]IPRf_N^N(0)\}. \tag{38}$$

To appreciate the resummation that is done later, we summarize the results for terms up to $j=5$.

$j=0$:

$$f(r,p,0);$$

$j=1$:

$$\frac{t}{1!}[L_of(r,p,0) + IPL_if_N^N(0)];$$

$j=2$:

$$\frac{t^2}{2!}[L_o^2f(r,p,o) + L_oIPL_if_N^N(0) + IPRf_N^N(0)];$$

$j=3$:

$$\frac{t^3}{3!}[L_o^3f(r,p,0) + \Delta L_of(r,p,0) + (L_o^2 + \Delta)IPL_if_N^N(0) + (L_o + \Delta L_o^{-1})IPRf_N^N(0)];$$

$j=4$:

$$\frac{t^4}{4!}[L_o^4f(r,p,0) + (L_o\Delta L_o + \Delta L_o^2 + \Delta L_o^{-1}\Delta L_o)f(r,p,0) + (L_o^3 + L_o\Delta + \Delta L_o + \Delta L_o^{-1}\Delta)IPL_if_N^N(0)$$
$$+ (L_o^2 + L_o\Delta L_o^{-1} + \Delta + \Delta L_o^{-1}\Delta L_o^{-1})IPRf_N^N(0)];$$

$j=5$:

$$\frac{t^5}{5!}[L_o^5f(r,p,0) + (L_o^2\Delta L_o + L_o\Delta L_o^2 + L_o\Delta L_o^{-1}\Delta L_o + \Delta L_o^3 + \Delta^2 L_o + \Delta L_o^{-1}\Delta L_o^2 + \Delta L_o^{-1}\Delta L_o^{-1}\Delta L_o)f(r,p,0)$$
$$+ (L_o^4 + L_o^2\Delta + L_o\Delta L_o + L_o\Delta L_o^{-1}\Delta + \Delta L_o^2 + \Delta^2 + \Delta L_o^{-1}\Delta L_o + \Delta L_o^{-1}\Delta L_o^{-1}\Delta)IPL_if_N^N(0)$$
$$+ (L_o^3 + L_o^2\Delta L_o^{-1} + L_o\Delta + L_o\Delta L_o^{-1}\Delta L_o^{-1} + \Delta L_o + \Delta^2 L_o^{-1} + \Delta L_o^{-1}\Delta + \Delta L_o^{-1}\Delta L_o^{-1}\Delta L_o^{-1})IPRf_N^N(0)].$$

By the procedure described above, all the terms may be written. It is then possible to regroup the terms, collect them, and finally write a resummation. We write the result of such a regrouping and summation:

$$f(r,p,t)=\sum_{j=0}^{\infty}\frac{t^j}{j!}L_o^j f(r,p,0)+\sum_{j=1}^{\infty}\frac{t^j}{j!}L_o^{j-1}IPL_i f_N^N(0)+\sum_{j=2}^{\infty}\frac{t^j}{j!}L_o^{j-2}IPRf_N^N(0)+\sum_{q=3}^{\infty}\left[\sum_{j=q}^{\infty}\frac{t^j}{j!}L_o^{j-q}S_q f(r,p,0)+\sum_{j=q}^{\infty}\frac{t^j}{j!}L_o^{j-q}T_q IPL_i f_N^N(0)+\sum_{j=q}^{\infty}\frac{t^j}{j!}L_o^{j-q}U_q IPRf_N^N(0)\right], \tag{39}$$

where

$$\begin{aligned}
S_3&=\Delta L_o,\\
S_4&=\Delta L_o^2+\Delta L_o^{-1}\Delta L_o=\Delta L_o(L_o+L_o^{-1}\Delta)L_o,\\
S_5&=\Delta L_o^3+\Delta^2 L_o+\Delta L_o^{-1}\Delta L_o^2+\Delta L_o^{-1}\Delta L_o^{-1}\Delta L_o^{-1}=\Delta L_o(L_o+L_o^{-1}\Delta)^2 L_o,\\
S_q&=\Delta L_o(L_o+L_o^{-1}\Delta)^{q-3}L_o,\\
T_3&=\Delta,\\
T_4&=\Delta L_o+\Delta L_o^{-1}=\Delta(L_o+L_o^{-1}\Delta),\\
T_5&=\Delta L_o^2+\Delta^2+\Delta L_o^{-1}\Delta L_o+\Delta L_o^{-1}\Delta L_o^{-1}\Delta=\Delta(L_o+L_o^{-1}\Delta)^2,\\
T_q&=\Delta(L_o+L_o^{-1}\Delta)^{q-3},\\
U_3&=\Delta L_o^{-1},\\
U_4&=\Delta+\Delta L_o^{-1}\Delta L_o^{-1}=\Delta(L_o+L_o^{-1}\Delta)L_o^{-1},\\
U_5&=\Delta L_o+\Delta^2 L_o^{-1}+\Delta L_o^{-1}\Delta+\Delta L_o^{-1}\Delta L_o^{-1}\Delta L_o^{-1}=\Delta(L_o+L_o^{-1}\Delta)^2 L_o^{-1},\\
U_q&=\Delta(L_o+L_o^{-1}\Delta)^{q-3}L_o^{-1}.
\end{aligned} \tag{40}$$

It is easy to surmise the formula for all S_q, T_q, U_q. Note that each of the above operators contains second derivatives with respect to momenta from the left, an important property that we will exploit later.

Following the technique in Refs. [1–3], we rewrite the infinite sums in j as telescoping time integrals. Then we use Eqs. (29)–(39) to simplify $IPL_i f_N^N(0)$ and $IPRf_N^N(0)$ to get

$$\begin{aligned}
f(r,p,t)&=f(r-pt/m,p,0)+n_0\int_0^t ds_1\ \exp(s_1L_o)\int dr'\ \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}f_2^1(r,r',p,0)\\
&-\int_0^t ds_1\int_0^t ds_2\ \exp(s_2L_o)n_0\int dr'\ \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}\left(\frac{p}{m}\frac{\partial}{\partial r}f_2^1(r,r',p,0)\right)\\
&-\int_0^t ds_1\int_0^t ds_2\ \exp(s_2L_o)n_0\int dp'\int dr'\ \frac{\partial V(r-r')}{\partial}\frac{\partial}{\partial p}\left(\frac{p'}{m}\frac{\partial}{\partial r'}f_2^2(r,r',p,p',0)\right)\\
&+n_0\int_0^t ds_1\int_0^t ds_2\ \exp(s_2L_o)\int dr'\left(\frac{\partial V(r-r')}{\partial r}\right)^2\left(\frac{\partial^2}{\partial p^2}f_2^1(r,r',p,0)\right)\\
&+n_0^2\int_0^t ds_1\int_0^t ds_2\ \exp(s_2L_o)\int dr'\int dr''\ \frac{\partial V(r-r')}{\partial r}\frac{\partial V(r-r'')}{\partial r}\left(\frac{\partial^2}{\partial p^2}f_3^1(r,r',r'',p,0)\right)\\
&\times\sum_{q=3}^{\infty}\int_0^t ds_1\int_0^{t_1}ds_2\cdots\int_0^{s_{q-1}}ds_q\ \exp(s_qL_o)\Bigg[S_qf(r,p,0)+n_0T_q\int dr'\ \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}f_2^1(r,r',p,0)\\
&-n_0U_q\int dr'\ \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}\left(\frac{p}{m}\frac{\partial}{\partial r}f_2^1(r,r',p,0)\right)-n_0U_q\int dp'\int dr'\ \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}\left(\frac{p'}{m}\frac{\partial}{\partial r'}f_2^2(r,r',p,p',0)\right)\\
&+n_0U_q\int dr'\left(\frac{\partial V(r-r')}{\partial r}\right)^2\left(\frac{\partial^2}{\partial p^2}f_2^1(r,r',p,0)\right)\\
&+n_0^2U_q\int dr'\int dr''\ \frac{\partial V(r-r')}{\partial r}\left(\frac{\partial V(r-r'')}{\partial r}\frac{\partial^2}{\partial p^2}f_3^1(r,r',r'',p,0)\right)\Bigg].
\end{aligned} \tag{41}$$

The simplest sum yields a displacement of $f(r,p,0)$ into $f(r-pt/m,p,0)$, simple ballistic propagation.

Insofar as S_k, T_k, and U_k may be written down, Eq. (41) is exact, and uses only one assumption, that the full distribution function possesses exchange symmetry between the $N-1$ particles whose momenta and coordinates are projected out. Curiously enough, the particle we are looking at, labeled by the index 1, could possess properties different from the $N-1$ others. This is an interesting possibility that could be explored much later. Note that for simplicity, we have now put $L_o=-(p/m)\partial/\partial r$.

Now we clarify the normalization needed to make all our variables consistent with conventional definitions. From the definitions of the projected probability distributions, we have

$$f_1^N(r_1,p_1,p_2,\ldots,p_N,t) = P_1 f_N^N(r_1,r_2,\ldots,r_N,p_1,p_2,\ldots,p_N,t) = \frac{1}{\Omega^{N-1}}\int dr_2\cdots dr_N f_N^N, \tag{42}$$

where Ω is the volume of the system, which up until this point we consider finite. As defined earlier, the subscripts refer to the coordinate label, while the superscripts refer to the momentum labels. There are N particle momenta, and 1 particle coordinate labels in the single-particle projected distribution function. Similarly, using the same notation, we have

$$f_2^N(r_1,r_2,p_1,p_2,\ldots,p_N,t) = f_2^N(r_1,r_2,p^N,t) = P_2 f_N^N(t) = \frac{1}{\Omega^{N-2}}\int dr_3 dr_4\cdots dr_N f_N^N(t), \tag{43}$$

$$f_3^N(r_1,r_2,r_3,p_1,p_2,\ldots,p_N,t) = f_3^N(r_1,r_2,r_3,p^N,t) = P_3 f_N^N(t) = \frac{1}{\Omega^{N-3}}\int dr_4 dr_5\cdots dr_N f_N^N(t). \tag{44}$$

The normalization of the momentum distribution functions, for one or many particles, do not need much attention. For example, we simply define the one-particle momentum probability as $\int dp\ \varphi(p,t)=1$ so that the momentum distribution function is dimensionally inverse momentum. However, to see the effect of the projection operators in the normalization of the density probabilities, consider the result of integration over the remaining momentum in Eq. (39), and use the definition of $\Delta=b\nabla_p^2$,

$$\begin{aligned} f(r,t) = {}& \int dp\ f(r-pt/m,p,0) + n_0\gamma\int dp\int_0^t ds_1\ \exp(s_1L_o)\int dr'\ \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p} f_2^1(r,r',p,0) \\ & - n_0\gamma\int dp\int_0^t ds_1\int_0^{s_1} ds_2\ \exp(s_2L_o)\int dr'\ \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}\left(\frac{p}{m}\frac{\partial}{\partial r} f_2^1(r,r',p,0)\right) \\ & - n_0\gamma\int dp\int_0^t ds_1\int_0^{s_1} ds_2\ \exp(s_2L_o)\int dp'\int dr'\ \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}\left(\frac{p'}{m}\frac{\partial}{\partial r'} f_2^2(r,r',p,p',0)\right) \\ & + n_0^2\gamma^2\int dp\int_0^t ds_1\int_0^{s_1} ds_2\ \exp(s_2L_o)\int dr'\int dr''\ \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}\left(\frac{\partial V(r-r'')}{\partial r}\frac{\partial}{\partial p} f_3^1(r,r',r'',p,0)\right) \\ & + n_0\gamma^2\int dp\int_0^t ds_1\int_0^{s_1} ds_2\ \exp(s_2L_o)\int dr'\ \frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p}\left(\frac{\partial V(r-r')}{\partial r}\frac{\partial}{\partial p} f_2^1(r,r',p,0)\right), \end{aligned} \tag{45}$$

where we have inserted the coupling parameter γ in front of the pair force as a bookkeeping tool. By integration over the momentum, the contribution of the rest of the infinite series vanishes for reasonable momentum distributions. Notice that in contrast to the BBGKY hierarchy [6] of equations—with its well-known closure problem—and the resulting need to know the N-particle distribution function to contract exactly to a one-particle distribution function by any of the techniques used so far [6], we provide closure by requiring that we know only the three-particle distribution function, and only at the initial condition at that. That we require only the three-particle distribution function instead of the full N-particle distribution function is due to the use of pair forces.

In this paper, we consider only factored initial conditions given by

$$f_3^N = f_1 f_1 f_1 \prod_{i=1}^{N} \varphi(p_i) \tag{46}$$

and use the spatial normalization

$$\int dr^N f_N = \int dr_1\int dr_2\cdots\int dr_N f_N(r_1,r_2,\ldots,r_N) = 1 \tag{47}$$

to give from Eqs. (41)–(43) the following conventional spatial density distributions:

$$\rho(r_1)=\int dr^{N-1}f_N(r^N)=\int dr_2\cdots dr_N f_N(r_1,r_2,\ldots,r_N)$$
$$=\Omega^{N-1}f_1(r_1), \tag{48}$$

$$\rho(r_1,r_2)=\int dr^{N-2}f_N(r^N)=\int dr_3\cdots dr_N f_N(r_1,r_2,\ldots,r_N)$$
$$=\Omega^{N-2}f_2(r_1,r_2), \tag{49}$$

$$\rho(r_1,r_2,r_3)=\int dr^{N-3}f_N(r^N)$$
$$=\int dr_4\cdots dr_N f_N(r_1,r_2,\ldots,r_N)$$
$$=\Omega^{N-3}f_3(r_1,r_2,r_3). \tag{50}$$

Using the above probability distributions, we may rewrite Eq. (44) as

$$\begin{aligned}\rho(r,t)=&\int dp\ \rho(r-pt/m,0)\varphi(p,0)+n_0\gamma\Omega\int dp\int_0^t ds_1\ \exp(s_1L_o)\int dr'\ \frac{\partial V(r-r')}{\partial r}\ \frac{\partial}{\partial p}\ \rho(r,0)\rho(r',0)\varphi(p,0)\\
&-n_0\gamma\Omega\int dp\int_0^{s_1}ds_1\int_0^t ds_2\ \exp(s_2L_o)\int dr'\ \frac{\partial V(r-r')}{\partial r}\ \frac{\partial}{\partial p}\left(\frac{p}{m}\ \frac{\partial}{\partial r}\ \rho(r,0)\rho(r',0)\varphi(p,0)\varphi(p',0)\right)\\
&-n_0\gamma\Omega\int dp\int_0^{s_1}ds_1\int_0^t ds_2\ \exp(s_2L_o)\int dp'\int dr'\ \frac{\partial V(r-r')}{\partial r}\ \frac{\partial}{\partial p}\left(\frac{p'}{m}\ \frac{\partial}{\partial r'}\ \rho(r,0)\rho(r',0)\varphi(p,0)\right)\\
&+n_0^2\gamma^2\Omega^2\int dp\int_0^{s_1}ds_1\int_0^t ds_2\ \exp(s_2L_o)\\
&\times\int dr'\int dr''\ \frac{\partial V(r-r')}{\partial r}\ \frac{\partial}{\partial p}\left(\frac{\partial V(r-r'')}{\partial r}\ \frac{\partial}{\partial p}[\rho(r,0)\rho(r',0)\rho(r'',0)\varphi(p,0)]\right)\\
&+n_0\gamma^2\int dp\int_0^t ds_1\int_0^{s_1}ds_2\ \exp(s_2L_o)\\
&\times\int dr'\ \frac{\partial V(r-r')}{\partial r}\ \frac{\partial}{\partial p}\left(\frac{\partial V(r-r')}{\partial r}\ \frac{\partial}{\partial p}\ \rho(r,0)\rho(r',0)\varphi(p,0)\right).\end{aligned} \tag{51}$$

Note that rigorously, from all previous work [2] and this one, we used the shortcut notation $n_0=(N-1)/\Omega$ and $n_0^2=(N-1)(N-2)/\Omega^2$ so that, in fact, the volume dependence of Eq. (50) is not there. What is constantly present is the total number of particles N. We thus get the interesting observation that one could define a new coupling parameter γN for a many-body system, and define a new weak-coupling limit $N\rightarrow\infty$, $\gamma\rightarrow 0$, which we only mention parenthetically, without using it in this work. With this explanation of the normalization convention, all our previous results are consistent with conventional normalization definitions. In Eq. (50), ρ is dimensionally inverse volume, making it consistent with the normalization given by Eqs. (41)–(43). This statement may be checked by dimensional analysis of Eq. (50). We will use ρ from now on.

To simplify Eq. (50), we may follow these steps: (1) Do the space integrations first over the forces. (2) Shift all variables using the equations

$$\exp\left(-\alpha p\ \frac{\partial}{\partial r}\right)F(r,r')=F(r-\alpha p,r')\exp\left(-\alpha p\ \frac{\partial}{\partial r}\right), \tag{52}$$

$$\exp\left(-\alpha p\ \frac{\partial}{\partial r}\right)\ \frac{\partial}{\partial p}\ H(r,p)=\left[\frac{\partial}{\partial p}\ H(r-\alpha p,p)+\alpha p\ \frac{\partial}{\partial r}\ H(r-\alpha p,p)\right]\times\exp\left(-\alpha p\ \frac{\partial}{\partial r}\right).$$

(3) Pull out, only when possible, all differentiations with respect to spatial coordinates to the left, paying attention to the commutativity of operations. (4) Do the integrations over time. (5) Perform the integration over momentum. In the above procedures, (4) and (5) may be interchanged.

IV. TIME EVOLUTION OF THE ONE-DIMENSIONAL GRAVITATIONAL GAS

To begin, we use the one-dimensional form of Eq. (51), and put

$$\frac{\partial V(r-r')}{\partial r}=\gamma\varepsilon(r-r'), \tag{53}$$

When Eq. (53) is substituted in Eq. (33), one gets

$$b=\frac{(N-1)\gamma^2}{L}\int dx'\,\varepsilon^2(x-x')=(N-1)\gamma^2$$

for any L (length) that takes the place of the volume Ω. In this work, we do not take the thermodynamic limit, so b is in fact finite.

Second, we assume an initial factored distribution of the form

$$\rho_3^1(r,r',r'',p,0)=\rho(r,0)\rho(r',0)\rho(r'',0)\varphi(p) \tag{54}$$

and consider the example of a ''petite-bang'' universe with a nonsingular explosion.

We use the initial Gaussian space and Maxwellian distribution

$$f(r,p,0)=\rho(r)\varphi(p), \tag{55}$$

where

$$\rho(r)=\sqrt{\frac{\alpha}{\pi}}\exp(-\alpha r^2),$$

$$\varphi(p)=\sqrt{\frac{\beta}{\pi}}\exp(-\alpha p^2)$$

and evaluate Eq. (51) term by term, labeling the terms by subscripts according to the order of appearance, $\rho=\rho_0+\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6$:

$$\rho_0=\sqrt{\frac{\alpha\beta}{\pi(\beta+\alpha t^2/m^2)}}\exp\left[-\alpha r^2\left\{1-\frac{\alpha t^2}{m^2(\beta+\alpha t^2/m^2)}\right\}\right], \tag{56}$$

$$\rho_1=\frac{(N-1)\gamma\beta m}{2\sqrt{\pi}}\left[\sqrt{\frac{\beta}{\pi}}\int dp\ \mathrm{erf}^2[\sqrt{\alpha}(r-pt/m)]\exp(-\beta p^2)-\mathrm{erf}^2[\sqrt{\alpha}r]\right], \tag{57}$$

$$\begin{aligned}\rho_2=&-\frac{4\alpha(N-1)\gamma}{m}\sqrt{\frac{\alpha\beta}{\pi^2}}\int_0^t ds_1\int_0^{s_1}ds_2\int dp(1-2\beta\rho^2)\exp[-\beta\rho^2](r-ps_2/m)\mathrm{erf}[\sqrt{\alpha}(r-ps_2/m)]\\&\times\exp[-\alpha(r-ps_2/m)^2]\\=&\frac{4(N-1)\gamma\sqrt{\alpha\beta}}{\pi}\int dp\left\{\frac{1}{2}\left[\mathrm{erf}(\sqrt{\alpha}r)\exp(-\alpha r^2)t-\int_0^t ds_1\ \mathrm{erf}(\sqrt{\alpha}[r-ps_1/m])\exp(-\alpha[r-ps_1/m]^2)\right]\right.\\&\left.+\frac{1}{\sqrt{2}}\left[\int_0^t ds_1\ \mathrm{erf}(\sqrt{\alpha}[r-ps_1/m])-\mathrm{erf}(\sqrt{\alpha}r)t\right]\right\}(1-2\beta p^2)\exp(-\beta p^2),\end{aligned} \tag{58}$$

$$\begin{aligned}\rho_3=&\frac{(N-1)(N-2)\gamma^2\sqrt{\alpha\beta}}{\pi}\int_0^t ds_1\int_0^{s_1}ds_2\int dp[2\beta^2p^2-1]\mathrm{erf}^2[\sqrt{\alpha}(r+ps_2/m)]\exp[-\alpha(r+ps_2/m)^2]\exp(-\beta p^2)\\=&\frac{(N-1)(N-2)\gamma^2 m}{6}\sqrt{\frac{\beta}{\pi}}\int dp\ \frac{(2\beta^2p^2-1)}{p}\exp(-\beta\rho^2)\left\{[\mathrm{erf}(\sqrt{\alpha}r)]^3t-\int_0^t ds_1[\mathrm{erf}(\sqrt{\alpha}[r+ps_1/m])]^3\right\},\end{aligned} \tag{59}$$

$$\rho_4=0, \tag{60}$$

$$\begin{aligned}\rho_5=&\frac{n_0^2\gamma^4\Omega^2}{3}\int dp\sqrt{\frac{\beta}{\pi}}(2\beta^2p^2-\beta)e^{-\beta p^2}\otimes\int_0^t ds_1\left\{\mathrm{erf}^3\left[\sqrt{\frac{\alpha}{\pi^3}}[-\sqrt{\pi}e^{-r^4}+\pi r^{-r^2}\ \mathrm{erf}(r)]\right]\right.\\&\left.-\mathrm{erf}^3\left[\sqrt{\frac{\alpha}{m^2\pi^3}}[-\sqrt{\pi}me^{-r^4}+\pi mre^{-r^2}\ \mathrm{erf}(r)]+\pi^{3/2}ps_1\right]\right\},\end{aligned} \tag{61}$$

$$\begin{aligned}\rho_6=&n_0^2\gamma^4\Omega^2\int dp(2\beta^2p^2-\beta)\sqrt{\frac{\beta}{\pi}}e^{-\beta p^2}\frac{m}{\alpha\sqrt{\pi}p^2}\left\{-\sqrt{\pi}\alpha t(p-mr/t)\mathrm{erf}\sqrt{\alpha}(r-pt/m)+m\sqrt{\alpha}(e^{-\alpha(r-pt/m)^2}-e^{-\alpha r^2})\right.\\&\left.+\alpha\sqrt{\pi}(pt-mr)\mathrm{erf}(\sqrt{\alpha}r)\right\}.\end{aligned} \tag{62}$$

Note that as $t \to \infty$, there is an irreversible approach to uniformity. It is for this reason that we introduce the concept of semiergodicity, ergodic only in spatial coordinates. In cosmological terms, this ''petite universe'' is an open universe. We have found this to be true for other initial momentum distributions. Exact statements about the behavior of the momentum distribution for this system is not so straightforward and will be treated elsewhere. Nevertheless, despite this semiergodicity, the short-time behavior is quite interesting. Microstructures naturally develop in time; this is an intriguing hint as to the development of substructures in three-dimensional gravitational systems [7]. These microstructures may be seen by expanding the time integrand in Taylor series, then integrating the results term by term. For example, up to second order in time, we get the following:

$$\rho_1 \approx \frac{2(N-1)\gamma\alpha t^2}{m\sqrt{\beta}}\left[\exp(-\alpha^2 r^4)/\pi - \sqrt{\alpha}r\ \mathrm{erf}(\sqrt{\alpha}r) \times \exp(-\alpha r^2)/\sqrt{\pi}\right], \tag{63}$$

which demonstrates the growth of the substructure. We postpone a more exhaustive numerical study of the formation and development of substructures or microstructures found elsewhere [8] and in this example to later work [9]. The substructures that we have so far found by starting with our analytic treatment are all irreversible.

V. SUMMARY AND PROSPECTS

We have introduced a formula for the time evolution of the density of an arbitrary classical many-body system using the method of iterated projections. In its final form, the formula no longer has any trace of projection operators, and may be immediately used for many physical applications. As a first test case, we have used it for a favorite toy system in stellar dynamics, providing some insight into the evolution of a one-dimensional gravitational system. In the course of this study, we introduce the concept of ''semiergodicity,'' pointing to some examples of irreversible systems that exemplify a Prigogine conjecture [10]. It would be interesting to see if a ''semiergodic'' system as introduced in this paper is in fact fully ergodic. That might offer some interesting consequence for the proof that some systems are indeed ergodic. In another work, we will describe the time evolution of the momentum distribution function; only at that point could one demonstrate the consistency of our results with well-known virial theorems in stellar dynamics [11].

We have elsewhere [12] applied our time evolution equation and used it to resolve some difficulties on the convergence of transport coefficients. We now raise the possibility of using our time evolution equation for three-dimensional problems, in stellar dynamics, in particular. Among such studies will be the time evolution of pair correlations, using projection operators that integrate all but two spatial coordinates. Such an approach may be applied to the evolution of binary stars in globular clusters [13]. But as the specification of the initial conditions affords us the freedom to vary the problem that we address, we think that there will be many more applications of our time evolution equation to many-body physics.

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